Exercise sheet 1

Monday 5th January 2026

Examples and properties of minimal surfaces

Exercise 1: Minimal catenoid in $\mathbb{R}^3$

1. (*Easy*) Let $\Sigma = \{(f(t)\cos(\theta), f(t)\sin(\theta), t) \,|\, t \in \mathbb{R}, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ be a surface of revolution in $\mathbb{R}^3$, for some function $f : \mathbb{R} \to (0, +\infty)$. Compute the first and second fundamental form, and the mean curvature, of Σ.

2. (*Easy*) Check that imposing $H = 0$ is equivalent to an ODE on f, and check that $f(t) = \cosh(t)$ is a solution of the ODE.

Exercise 2: The Clifford torus in $\mathbb{S}^3$

1. (*Easy*) Show that, in $\mathbb{S}^3 = \{x_1^2+x_2^2+x_3^2+x_4^2 = 1\}$ the Clifford torus $\{x_1^2+x_2^2 = x_3^2+x_4^2 = 1/2\}$ is a minimal surface.

2. (*Easy*) Show that its first fundamental form is flat and its area equals $2\pi^2$.

Comment: the smallest area closed minimal surface in $\mathbb{S}^3$ is (up to isometry) the totally geodesic sphere $\mathbb{S}^2 = \{x_4 = 0\}$. The Willmore Conjecture, proved by Marques and Neves in 2014, asserts that the second one (and the only one achieving the value $2\pi^2$, up to isometry) is the Clifford torus. Also, the Clifford torus is (up to isometry) the only minimal embedded torus in $\mathbb{S}^3$: this is the statement of the Lawson Conjecture, proved by Brendle in 2013.

Exercise 3: Minimal surface equation in $\mathbb{R}^3$

1. (*Easy*) Let $\Sigma \subset \mathbb{R}^3$ be the graph of a function $u : \Omega \to \mathbb{R}$, for Ω an open set in $\mathbb{R}^2$. Compute the first fundamental form, the area form, the second fundamental form and the mean curvature of Σ with respect to the graphical parameterization $(x, y) \mapsto (x, y, u(x, y))$.

2. (*Easy*) Show that Σ is minimal if and only if u satisfies the equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0\ . \tag{1}$$

3. (*Medium*) Show that (assuming Ω and u bounded) the area of Σ equals

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |Du|^2}\,dxdy\ .$$

   Then prove that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(u + tv) = 0$$

   for all smooth bounded $v : \Omega \to \mathbb{R}$ if and only if the following equation (the Euler-Lagrange equation of $\mathcal{A}$) holds:

$$\mathrm{div}_{\mathbb{R}^2}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0\ . \tag{2}$$

4. (*Medium*) Check that (2) is actually equivalent to the minimal surface equation (1).

Comment: a consequence of the exercise is that, for (smooth!) graphs over $\mathbb{R}^2$ in $\mathbb{R}^3$ (and even for graphs in $\mathbb{R}^{n+1}$ over $\mathbb{R}^n$), being minimal (i.e. critical point of the area with respect to any variation) is actually equivalent to being a critical point of the area with respect to (a priori) only graphical variations, also called outer variations. The De Giorgi-Nash-Moser Theorem shows that this is also true for *Lipschitz* functions u that are critical points of the area functional under outer variations. The result was recently extended in any codimension by Hirsch-Mooney-Tione, who solved a conjecture of Lawson-Osserman and proved that Lipschitz weak solutions are C^2.

Exercise 4: Geodesic equation

The purpose of this exercise is to warm-up for the next one, by showing that geodesics on Riemannian manifolds are precisely the critical points of the length functional.

1. (*A bit difficult*) Show that, given a curve $\gamma : I \to (N,h)$ for $I = [a,b]$ or S^1, (N,h) a Riemannian manifold, parameterized by arclength, and a smooth variation $\Gamma : [a,b] \times (-\epsilon, \epsilon)$ of γ with fixed endpoints (i.e. $\Gamma(t,0) = \gamma(t)$ and, if I is a closed interval, $\Gamma(a,s) = \gamma(a)$, $\Gamma(b,s) = \gamma(b)$ for all s), the following formula holds:
$$\frac{d}{ds}\bigg|_{s=0} \mathcal{L}(\gamma_s) = -\int h(V(\gamma(t)), \nabla^h_{\gamma'(t)}\gamma'(t))dt \ ,$$
where $\gamma_s(\cdot) = \Gamma(\cdot, s)$, $V(\gamma(t)) = d\Gamma(t,s)/ds|_{s=0}$ and $\mathcal{L}$ denotes the length of a curve.

2. (*Medium*) Deduce that γ is a geodesic (i.e. $\nabla^h_{\gamma'(t)}\gamma'(t)) = 0$) if and only if it is a critical point of the length under smooth variations of γ (with fixed endpoints, if I is an interval).

Exercise 5: Minimal surfaces are critical points of the area

The goal is now to show that minimal immersions are precisely the critical points of the area functional.

1. (*More difficult*) Show that, given an immersion $\iota : M \to (N,h)$ for (N,h) a Riemannian manifold and a smooth compactly supported variation $\Gamma : M \times (\epsilon, \epsilon) \to M$ such that $\Gamma(x,0) = \iota$, the following formula holds:
$$\frac{d}{ds}\bigg|_{s=0} \mathrm{dVol}_{g_s} = -h(V,H)\mathrm{dVol}_g$$
where $V = d\Gamma(\cdot,s)/ds|_{s=0}$, H is the mean curvature of ι, g_s is the first fundamental form of $x \mapsto \Gamma(x,s)$, and recall that, in coordinates, $\mathrm{dVol}_g = \sqrt{\det g}dx_1 \wedge \ldots \wedge dx_n$. [Hint: to simplify the computation, take normal coordinates around a given point x, so that $g = g_0$ is expressed as δ_{ij} at x. First show that the derivative of dVol_{g_s} equals $\mathrm{tr}(dg_s/ds|_{s=0})$, and finally show that the latter quantity is equal to $-h(V,H)$.]

2. (*Medium*) As a consequence, if $\mathcal{A}$ denotes the area (or volume, if $\dim M > 2$) of an immersion,
$$\frac{d}{ds}\bigg|_{s=0} \mathcal{A}(\iota_s) = -\int_M h(V,H)\mathrm{dVol}_g \ .$$
Deduce that ι is minimal (i.e. $H = 0$) if and only if it is a critical point of $\mathcal{A}$ under compactly supported variations.

Exercise 6: Minimal surfaces are conformal harmonic maps

Recall that, given a smooth function f on a Riemannian manifold (M,g) taking values in $\mathbb{R}$, the Hessian of f is the $(0,2)$-tensor $\nabla^2 f(X,Y) := (\nabla^g_X df)(Y)$. The Laplace-Beltrami operator is defined as $\Delta^g f := \mathrm{tr}_g \nabla^2 f$.

1. (*Easy*) Extending the definition of Hessian to a function $f : \Sigma \to \mathbb{R}^n$ coordinate by coordinate, show that the second fundamental form of an immersion $\iota : \Sigma \to \mathbb{R}^3$ is equal to the Hessian of ι with respect to the first fundamental form $g = \iota^* g_{\mathbb{R}^3}$, and that the mean curvature is equal to $\Delta^g \iota$.

2. (*Easy*) Deduce that an immersion $\iota = (\iota_1, \iota_2, \iota_3) : \Sigma \to \mathbb{R}^3$ is minimal if and only if each component ι_i is harmonic with respect to the first fundamental form $g = \iota^* g_{\mathbb{R}^3}$, that is, $\Delta^g \iota_i = 0$.

3. (*Medium*) Prove that a function $f : (M,g) \to \mathbb{R}$ (or $\mathbb{R}^n$) is harmonic if and only if it is a critical point of the energy
$$\mathcal{E}(f) := \frac{1}{2}\int_M \|df\|^2_g \mathrm{dVol_g}$$
under compactly supported variations of f, where $\|df\|^2_g = |df(e_1)|^2 + |df(e_2)|^2$ for a g-ortonormal frame (e_1, e_2). Show that if $\dim M = 2$, then $\mathcal{E}(f)$ only depends on the conformal class of g.

4. (*Medium*) Show that, for a surface Σ and an immersion $\iota : \Sigma \to \mathbb{R}^3$, $\mathcal{A}(\iota) \leq \mathcal{E}(\iota)$, with equality if and only if ι is conformal (i.e. if $\iota^* g_{\mathbb{R}^3} = e^{2\varphi} g$). Give an alternative proof of the fact that ι is minimal if and only if it is harmonic with respect to (any metric conformal to) the first fundamental form on Σ.

Comment: the above results still hold if one replaces $\mathbb{R}^3$ with any Riemannian manifold. There is a notion of harmonic maps between Riemannian manifolds (M, g) and (N, h), which is equivalent to the vanishing of the tension $\tau(f)(X, Y) = \nabla^h_X df(Y) - df(\nabla^g_X Y)$. Minimal immersions are then precisely the conformal harmonic maps.

Exercise 7: Minimal surfaces are locally area minimizing

The goal here is to show that minimal surfaces are precisely the surfaces that locally minimize area among disks (not only critical points, but actually minima, in a small neighbourhood of every point). This uses ideas from the theory of calibrations. For simplicity, we do this when the ambient space is $\mathbb{R}^3$.

- We say that a 2-form ω on (N, h) is a *calibration* if it satisfies:

  (a) $d\omega = 0$

  (b) $|\omega(E_1, E_2)| \leq 1$ for every pair of h-orthonormal vectors (E_1, E_2).

  A surface Σ in N is *calibrated* if it satisfies:

  (c) $\omega(E_1, E_2) = 1$ for every pair of h-orthonormal vectors (E_1, E_2) tangent to Σ (equivalently, $\omega|_\Sigma = \mathrm{dVol}_\Sigma$).

- Given a minimal surface $\Sigma \subset \mathbb{R}^3$ and $p \in \Sigma$, up to an isometry, we assume that in a neighbourhood U of p,

$$\Sigma_\Omega := \Sigma \cap U = \{(x, y, u(x, y)) \mid (x, y) \in \Omega\}$$

  for Ω a (small) convex set.

1. (*Medium*) Assume that there exists a calibration ω in $\Omega \times \mathbb{R}$ such that Σ_Ω is calibrated. Using Stokes' Theorem, prove that the area of Σ_Ω is less than or equal to the area of any other "competitor" Σ' which is topologically a disk, contained in $\Omega \times \mathbb{R}$, with $\partial\Sigma' = \partial\Sigma_\Omega$. [Hint: don't forget to use all the properties that define a calibration.]

2. (*Medium*) Now prove that the area of Σ_Ω is less than or equal to the area of any other "competitor" Σ' which is topologically a disk (not necessarily contained in $\Omega \times \mathbb{R}$) with $\partial\Sigma' = \partial\Sigma_\Omega$. [Hint: use that the nearest point projection to a convex set decreases distances.]

- Now, to apply the previous two points, we construct a calibration in $\Omega \times \mathbb{R}$ that calibrates Σ_Ω. Let ν be the unit normal vector to Σ_Ω, extended to a vector field on $\Omega \times \mathbb{R}$ in such a way that it does not depend on the z-coordinate. Define

$$\omega(X, Y) = \mathrm{dVol}_{\mathbb{R}^3}(X, Y, \nu)$$

  where $\mathrm{dVol}_{\mathbb{R}^3} = \det$.

3. (*Easy*) Prove that ω satisfies properties (b) and (c).

4. (*Medium*) Compute ω in (x, y, z)-coordinates and prove that ω satisfies property (a) if and only if Σ_Ω is minimal. [Hint: the computation is simplified if one uses the expression (2) from Exercise 3 of the minimal graph equation.]